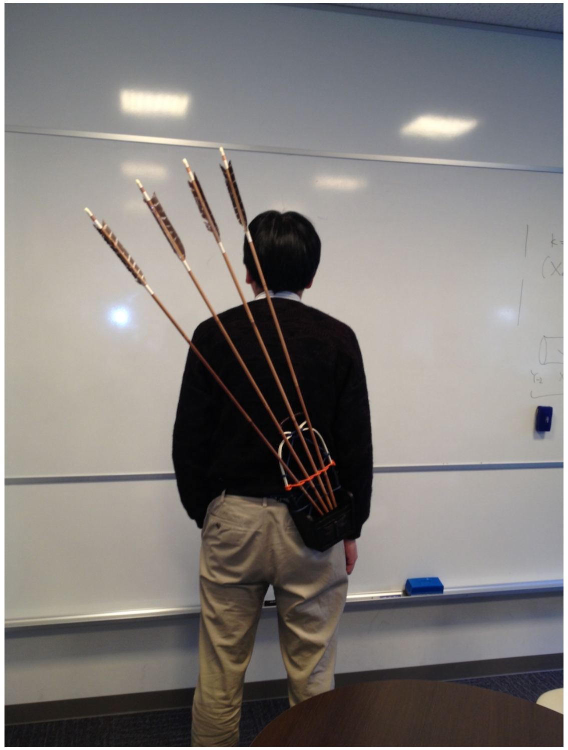
What is a quiver in Mathematics ?



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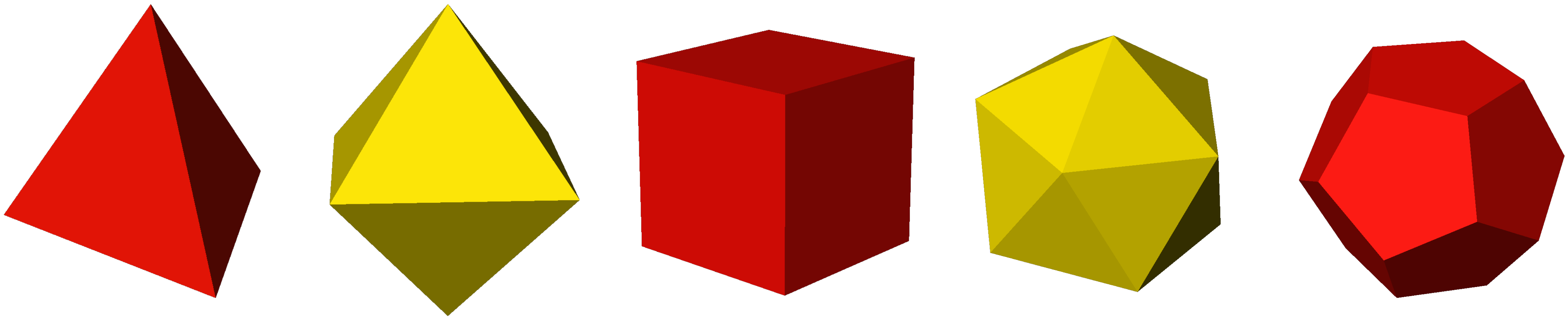
ICIAM Workshop

Vietnam Institute for Advanced Study in Mathematics (VIASM)

Before going to quiver, let me recall the classification of regular polyhedra.



tetrahedron octahedron hexahedron icosahedron dodecahedron

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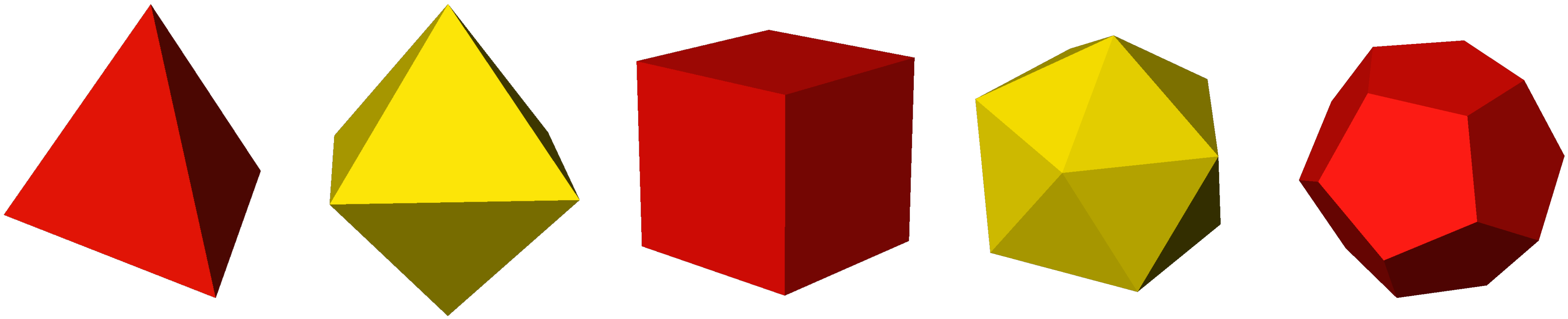

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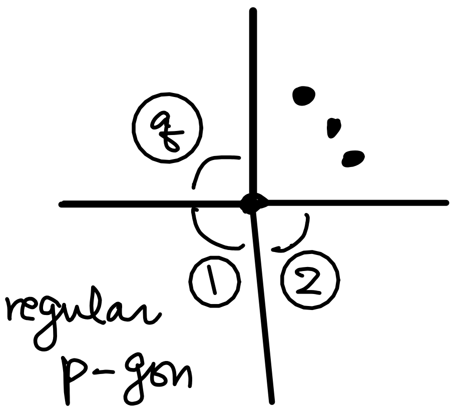


q = # of regular p-gons at each vertex

$$180 \times \frac{p-2}{p} \times q < 360 \iff \frac{1}{p} + \frac{1}{q} > \frac{1}{2}$$

$$\iff (p, q) = (3,3), (3,4), (4,3), (3,5), (5,3)$$

Quiver Q : (finite) oriented graph

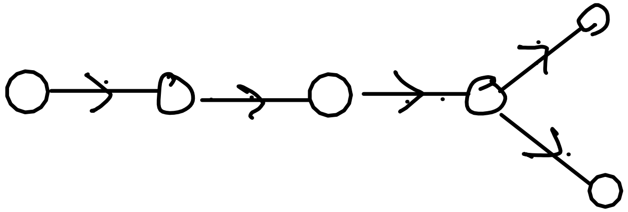
Q_0 = set of vertices

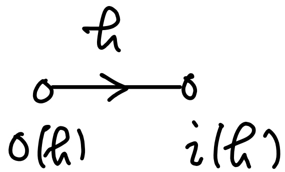
Q_1 = set of oriented edges

$$
\begin{tikzcd}
 & & & & \circ \\
\circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[ur] \arrow[dr] & \\
 & & & & \circ
\end{tikzcd}
$$

$$
\begin{tikzcd}
o(h) \arrow[r, "h"] & i(h)
\end{tikzcd}
$$

Quiver Q : (finite) oriented graph Q_0 = set of vertices
k : field (fix) Q_1 = set of oriented edges





Def. A **representation** of a quiver Q is

— a collection of vector spaces V_i $(i \in Q_0)$

— a collection of linear maps $A_h : V_{o(h)} \longrightarrow V_{i(h)}$ $(h \in Q_1)$

Def. A **homomorphism** from $\{(V_i, A_h)\}$ to $\{(V_i', A_h')\}$ is

— a collection of linear maps $P_i : V_i \longrightarrow V_i'$
such that

$$\begin{tikzcd} V_{o(h)} \arrow[r, "A_h"] \arrow[d, "P_{o(h)}"'] & V_{i(h)} \arrow[d, "P_{i(h)}"] \\ V'_{o(h)} \arrow[r, "A'_h"] & V'_{i(h)} \end{tikzcd}$$

commutes.

Example

(1) $\circ \longrightarrow \circ$ $\quad V_1 \xrightarrow{A} V_2$ $\qquad$ (2) $\circ\!\circlearrowleft$ $\quad V \circlearrowleft A$ $\qquad$ (3) $\circ \rightrightarrows \circ$ $\quad V_1 \underset{B}{\overset{A}{\rightrightarrows}} V_2$

Problem Classify ALL representations up to isomorphisms

Example (1) Classify A up to $A \sim PAQ^{-1}$ (P, Q : invertible matrices)

Ans :

$$A \sim \begin{bmatrix} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{\operatorname{rank} A} & 0 \\ 0 & \end{bmatrix}$$

Example (2) : Classify A up to $A \sim PAP^{-1}$ (P : invertible matrices)

Ans :

$$A \sim \left[\begin{array}{c|c|c} \begin{matrix} \lambda_1 & 1 & \\ & \ddots & 1 \\ & & \lambda_1 \end{matrix} & & \\ \hline & \begin{matrix} \lambda_2 & 1 & \\ & \ddots & 1 \\ & & \lambda_2 \end{matrix} & \\ \hline & & \ddots \end{array}\right]$$

Jordan normal form

assumption : over algebraically closed field

It is better to look at "building blocks"

Def – A direct sum representation : $V_i \oplus V_i'$ and $\begin{bmatrix} A_{a_i} & 0 \\ 0 & A'_{a_i} \end{bmatrix}$

– An indecomposable representation $\Leftrightarrow$ not isomorphic to a direct sum

Return back to Examples

(2) $V \circlearrowleft A$ indecomposable. $\quad A \sim \begin{bmatrix} x & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & x \end{bmatrix}$ Jordan cell

(1) $V_1 \xrightarrow{A} V_2$ indecomposable $\quad A \sim k \xrightarrow{1} k$

or $k \xrightarrow{0} 0$ $\quad$ 3 types

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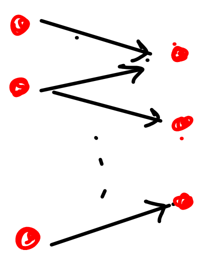
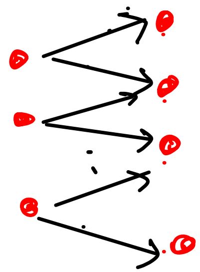
(3) (Kronecker) k^n  k^{n-1} or k^{n-1}  k^n or

$$\begin{bmatrix} x_1 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & x_1 \end{bmatrix}$$

$$k^n \rightrightarrows k^n$$

$$\begin{bmatrix} x_2 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & x_2 \end{bmatrix}$$

(x_1, x_2)

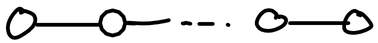
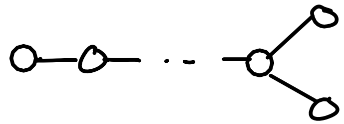
$\sim (\lambda x_1, \lambda x_2)$

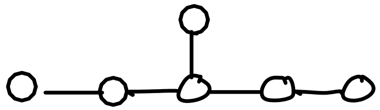
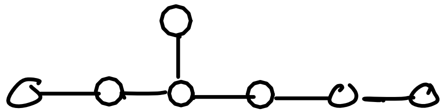
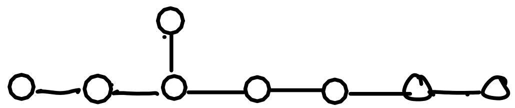
$[x_1 : x_2] \in \mathbb{P}^1(k)$

Theorom (Gabriel)

Q has only finitily many indecomposable represantations up to isom.

$\Longleftrightarrow$ The underlying unoriented graph is one of the followings :

A_n  D_n 

E_6  E_7  E_8 

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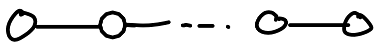
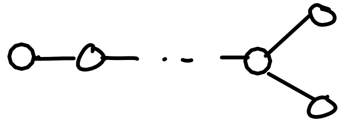
A_n  D_n 

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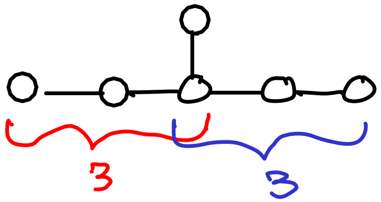

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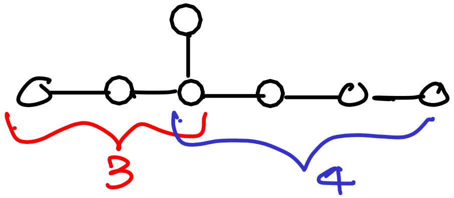

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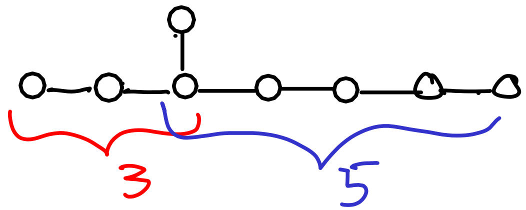
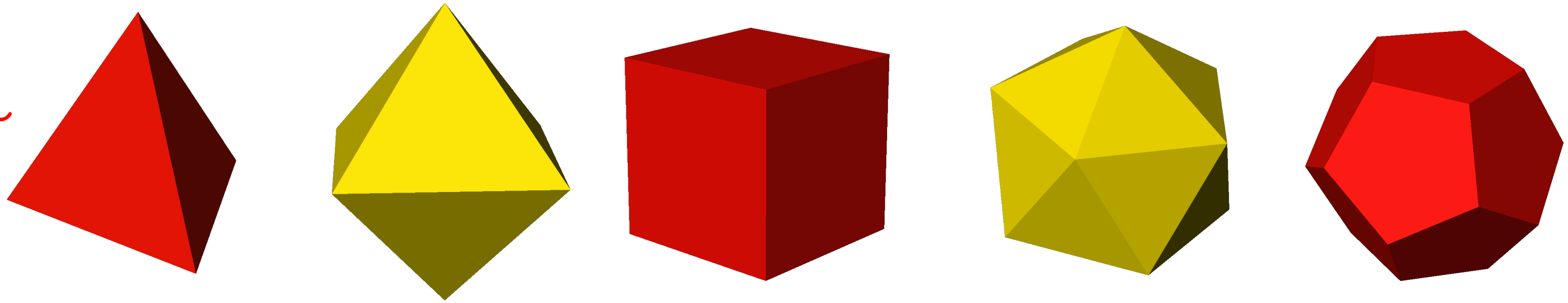


Relation to regular polyhedra



Example (Type A)

Indecomposable representations are $0 \to 0 \to \cdots \to 0 \to k \to k \to \cdots \to k \to 0 \to \cdots \to 0$

(For any orientations of arrows)

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This result is a basis of Persistent homology theory in topological data analysis.

Namely a sequence of topological spaces $\cdots \subset X_n \subset X_{n+1} \subset \cdots$ gives

$\cdots \longrightarrow H_i(X_n) \longrightarrow H_i(X_{n+1}) \longrightarrow \cdots$, a representation of type A quiver.

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What am I doing on quivers ?

Not finite type $\Longrightarrow$

- Consider the space of all (well-behaved) representations as a geometric object.
- Relation to Lie algebras.